Let us prove that for an n-dimensional function $f(x_1, x_2, \ldots, x_n)$, the directional second derivative $\frac{d^2}{dt^2} f(\vec{p} + t\vec{v})\Big|_{t=0} = (\vec{v})^T(H_p f)v$, where $H_p f$ is the hessian matrix of f at point $\vec{p}$.

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$H_p f = \begin{bmatrix} f_{x_1x_1}(\vec{p}) & f_{x_1x_2}(\vec{p}) & f_{x_1x_3}(\vec{p}) & \ldots & f_{x_1x_n}(\vec{p}) \\ f_{x_2x_1}(\vec{p}) & f_{x_2x_2}(\vec{p}) & f_{x_2x_3}(\vec{p}) & \ldots & f_{x_2x_n}(\vec{p}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\vec{p}) & f_{x_nx_2}(\vec{p}) & f_{x_nx_3}(\vec{p}) & \ldots & f_{x_nx_n}(\vec{p}) \end{bmatrix}$$

Let us find $(\vec{v})^T(H_p f)v$.

$$\begin{aligned}
(\vec{v})^T(H_p f)\vec{v} &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} f_{x_1x_1}(\vec{p}) & f_{x_1x_2}(\vec{p}) & f_{x_1x_3}(\vec{p}) & \ldots & f_{x_1x_n}(\vec{p}) \\ f_{x_2x_1}(\vec{p}) & f_{x_2x_2}(\vec{p}) & f_{x_2x_3}(\vec{p}) & \ldots & f_{x_2x_n}(\vec{p}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\vec{p}) & f_{x_nx_2}(\vec{p}) & f_{x_nx_3}(\vec{p}) & \ldots & f_{x_nx_n}(\vec{p}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} v_1 f_{x_1x_1}(\vec{p}) + v_2 f_{x_1x_2}(\vec{p}) + \ldots + v_n f_{x_1x_n}(\vec{p}) \\ v_1 f_{x_2x_1}(\vec{p}) + v_2 f_{x_2x_2}(\vec{p}) + \ldots + v_n f_{x_2x_n}(\vec{p}) \\ \vdots \\ v_1 f_{x_nx_1}(\vec{p}) + v_2 f_{x_nx_2}(\vec{p}) + \ldots + v_n f_{x_nx_n}(\vec{p}) \end{bmatrix} \\
&= v_1(v_1 f_{x_1x_1}(\vec{p}) + v_2 f_{x_1x_2}(\vec{p}) + \ldots + v_n f_{x_1x_n}(\vec{p})) \\
&+ v_2(v_1 f_{x_2x_1}(\vec{p}) + v_2 f_{x_2x_2}(\vec{p}) + \ldots + v_n f_{x_2x_n}(\vec{p})) \\
&\vdots \\
&+ v_n(v_1 f_{x_nx_1}(\vec{p}) + v_2 f_{x_nx_2}(\vec{p}) + \ldots + v_n f_{x_nx_n}(\vec{p}))
\end{aligned}$$

Let us find $\frac{d^2}{dt^2} f(\vec{p} + t\vec{v})\Big|_{t=0}$.

$$\begin{aligned} g(t) &= f(\vec{p} + t\vec{v}) \\ x_1(t) &= p_1 + tv_1 \\ x_2(t) &= p_2 + tv_2 \\ &\vdots \\ x_n(t) &= p_n + tv_n \end{aligned}$$

Note that for each x_i, the second derivative is zero.

$$\begin{aligned} \frac{d^2 x_1}{dt^2} &= 0 \\ \frac{d^2 x_2}{dt^2} &= 0 \\ &\vdots \\ \frac{d^2 x_n}{dt^2} &= 0 \end{aligned}$$

Keeping this in mind, and using the multivariable chain rule and the product rule, we can conclude the following.

$$\begin{aligned} \frac{d^2}{dt^2} f(\vec{p} + t\vec{v})\Big|_{t=0} &= \frac{d^2}{dt^2} g(t)\Big|_{t=0} \\ &= \frac{d}{dt}[g_{x_1}\frac{dx_1}{dt} + g_{x_2}\frac{dx_2}{dt} + \cdots + g_{x_n}]\Big|_{t=0} \\ &= \frac{d}{dt}[g_{x_1}\frac{dx_1}{dt}] + \frac{d}{dt}[g_{x_2}\frac{dx_2}{dt}] + \cdots + \frac{d}{dt}[g_{x_n}\frac{dx_n}{dt}]\Big|_{t=0} \\ &= g_{x_1}\frac{d^2x_1}{dt^2} + \frac{d}{dt}[g_{x_1}]\frac{dx_1}{dt} + g_{x_2}\frac{d^2x_2}{dt^2} + \frac{d}{dt}[g_{x_2}]\frac{dx_2}{dt} + \cdots + g_{x_n}\frac{d^2x_n}{dt^2} + \frac{d}{dt}[g_{x_n}]\frac{dx_n}{dt}\Big|_{t=0} \\ &= \frac{d}{dt}[g_{x_1}]\frac{dx_1}{dt} + \frac{d}{dt}[g_{x_2}]\frac{dx_2}{dt} + \cdots + \frac{d}{dt}[g_{x_n}]\frac{dx_n}{dt}\Big|_{t=0} \\ &= (g_{x_1x_1}\frac{dx_1}{dt} + g_{x_1x_2}\frac{dx_2}{dt} + \cdots + g_{x_1x_n}\frac{dx_n}{dt})\frac{dx_1}{dt} \\ &+ (g_{x_2x_1}\frac{dx_1}{dt} + g_{x_2x_2}\frac{dx_2}{dt} + \cdots + g_{x_2x_n}\frac{dx_n}{dt})\frac{dx_2}{dt} \\ &\vdots \\ &+ (g_{x_nx_1}\frac{dx_1}{dt} + g_{x_nx_2}\frac{dx_2}{dt} + \cdots + g_{x_nx_n}\frac{dx_n}{dt})\frac{dx_n}{dt}\Big|_{t=0} \end{aligned}$$

Then, we take note of the following.

$$\begin{aligned} g_{x_i x_j}(0) &= f_{x_i x_j}(\vec{p}) \\ \frac{dx_1}{dt} &= v_1 \\ \frac{dx_2}{dt} &= v_2 \\ &\vdots \\ \frac{dx_n}{dt} &= v_n \end{aligned}$$

We can use this to expand the following.

$$\begin{aligned} &(g_{x_1x_1}\frac{dx_1}{dt} + g_{x_1x_2}\frac{dx_2}{dt} + \cdots + g_{x_1x_n}\frac{dx_n}{dt})\frac{dx_1}{dt} \\ &+ (g_{x_2x_1}\frac{dx_1}{dt} + g_{x_2x_2}\frac{dx_2}{dt} + \cdots + g_{x_2x_n}\frac{dx_n}{dt})\frac{dx_2}{dt} \\ &\vdots \\ &+ (g_{x_nx_1}\frac{dx_1}{dt} + g_{x_nx_2}\frac{dx_2}{dt} + \cdots + g_{x_nx_n}\frac{dx_n}{dt})\frac{dx_n}{dt}\Bigg|_{t=0} \\ &= (f_{x_1x_1}(\vec{p})v_1 + f_{x_1x_2}(\vec{p})v_2 + \ldots + f_{x_1x_n}(\vec{p})v_n)v_1 \\ &+ (f_{x_2x_1}(\vec{p})v_1 + f_{x_2x_2}(\vec{p})v_2 + \ldots + f_{x_2x_n}(\vec{p})v_n)v_2 \\ &\vdots \\ &+ (f_{x_nx_1}(\vec{p})v_1 + f_{x_nx_2}(\vec{p})v_2 + \ldots + f_{x_nx_n}(\vec{p})v_n)v_n \end{aligned}$$

We note that this is equal to what we got when we expanded $(\vec{v})^T(H_pf)\vec{v}$, and thus can conclude the following.

$$\frac{d^2}{dt^2}[f(\vec{p}+t\vec{v})]\Bigg|_{t=0} = (\vec{v})^T(H_pf)\vec{v}$$